Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(02)00049-7

J. Appl. Maths Mechs, Vol. 66, No. 3, pp. 401–408, 2002

0021-8928/02/$—see front matter

NON-LINEAR RESONANCE EVOLUTIONARY EFFECTS IN THE MOTION OF A RIGID BODY ABOUT A FIXED POINT†

Yu. M. ZABOLOTNOV and V. V. LYUBIMOV

Samara

(*Received* 17 *March* 2000)

Evolutionary effects in the motion of a heavy dynamically symmetrical rigid body about a fixed point in cases close to the Lagrange case are investigated in the non-linear formulation. Evolutionary effects are due to perturbations acting on the body, namely, a small displacement of the centre of mass with respect to the axis of dynamic symmetry, a perturbing moment, constant in a connected system of coordinates, and dissipative moments. It is shown that the presence of perturbations leads to the existence of "attracting" or "repulsing" resonances, which determine the evolution of the system in non-resonance parts of the motion. The method of integral manifolds is used for singularly perturbed systems and the method of averaging. A qualitative representation of possible motions of a statically stable top in the phase plane, taking a lower-order resonance into account, is given. © 2002 Elsevier Science Ltd. All rights reserved.

The difficulties involved in investigating resonance phenomena in the motion of a heavy rigid body about a fixed point due to the action of various perturbations [1–4, etc.] is due mainly to the non-linearity of the equations of motion and the variability of the frequencies characterizing the motion. Below we use the method of integral manifolds [5–7] for an initial simplification of the system of equations of motion of the body. The basis of this approach is the presence in the system of dissipative moments, which lead to the fact that the trajectory of the system over time tends to a certain integral manifold [7]. Hence, the evolution of the system can be investigated by studying the behaviour of the system on an integral manifold. The motion over the integral manifold corresponds to the motions of a body close to precessional. To construct an approximate system of equations of motion of the body with respect to integral manifolds we use the method of separation of the motions for singularly perturbed systems [6, 7]. On the other hand, the motion over the integral manifold in this problem is described by a single-frequency system, where the fact that the frequency for the initial system vanishes leads to a lower-order resonance, which has the greatest effect on the motion of the body.

Particular attention is devoted below to non-resonant regions of the motion of the system, for which approximate evolutionary equations are obtained. It is shown that sometimes one must construct a second approximation of the averaging method. In this sense, the evolutionary motions investigated are analogous to secondary resonance effects [8]. The method developed here has its own sources, in which the obtaining of quasi-static and low-frequency equations of motion of a rigid body in a resisting medium were considered [9, 10]. The method of averaging was used in [3, 4] to investigate the motion of a body close to regular precession, but non-linear evolutionary resonance effects were not considered.

1. THE EQUATIONS OF MOTION OF A RIGID BODY

The positions of a rigid body when it rotates about a fixed point are given by the Euler angles ψ, θ, φ, which connect a fixed system of coordinates $OX_1Y_1Z_1$ and a moving system of coordinates $OXYZ$, rotating together with the body. The axial moments of inertia I_x, I_y and I_z correspond to a dynamically symmetrical body, $I_x = I_y = I > I_z$, while the position of the centre of mass is defined by the vector $\Delta\mathrm{r} = \Delta x\mathbf{i} + \Delta y\mathbf{j} + \Delta z\mathbf{k}$, specified in the system of coordinates $OXYZ$ connected with the body. In cases close to the Lagrange case, we usually assume $0 < (\Delta x^2 + \Delta y^2)^{1/2} \ll |\Delta z|$, where, without loss of generality, we can put $\Delta x = 0$, by choosing the position of the connected system of coordinates $OXYZ$ in an appropriate way. Henceforth we will refer the value of the axial moment of inertia I_z to the value I and retain the previous notation.

†*Prikl. Mat. Mekh.* Vol. 66, No. 3, pp. 410–417, 2002.

In addition to the moment of the gravity force, a perturbing moment

$$\Delta\mathbf{M} = \Delta M_x\mathbf{i} + \Delta M_y\mathbf{j} + \Delta M_z\mathbf{k}$$

constant in the connected system, acts on the body, as well as a small dissipative moment

$$\mathbf{M}^{\omega} = M_x^{\omega}\mathbf{i} + M_y^{\omega}\mathbf{j} + M_z^{\omega}\mathbf{k}; \quad M_u^{\omega} = m^{\omega_u}\omega_u, \quad u = x, y, z$$

where ω_u are the components of the angular velocity of the body in the system of coordinates $OXYZ$, and m^{ω_u} are constant non-positive coefficients, where, due to the symmetry of the body, $m^{\omega_x} = m^{\omega_y} = m^{\omega}$. Here and henceforth all the moments relate to the quantity I.

The equations of perturbed motion of the body about a fixed point can be written in a form convenient for using asymptotic methods [2, 9, 11], as follows:

$$\dot{Q} = M_{z1}, \quad I_z\dot{\omega}_z = M_z$$

$$\ddot{\theta} = M_{xn} + F_1(x,\theta)F_2(x,\theta), \quad \dot{\varphi} = \omega_z - \omega_\psi\cos\theta \tag{1.1}$$

$$M_{z1} = M_{yn}\sin\theta + M_z\cos\theta$$

$$M_{xn} = M_x\cos\varphi - M_y\sin\varphi, \quad M_{yn} = M_y\cos\varphi + M_x\sin\varphi$$

$$M_z = -G\Delta y\sin\theta\sin\varphi + \Delta M_z + m^{\omega_z}\omega_z$$

$$M_x = G\Delta y\cos\theta - G\Delta z\sin\theta\cos\varphi + \Delta M_x + m^{\omega}\omega_x$$

$$M_y = G\Delta z\sin\theta\sin\varphi + \Delta M_y + m^{\omega}\omega_y$$

$$\omega_\psi = \dot{\psi} = F_1(x,\theta), \quad F_1(x,\theta) = (Q - I_z\omega_z\cos\theta)/\sin^2\theta$$

$$F_2(x,\theta) = (Q\cos\theta - I_z\omega_z)/\sin\theta$$

Here Q is the projection, referred to I, of the kinetic moment of the body onto the vertical OZ_1, G is the modulus of the gravity force of the body, referred to I, $x = (Q, \omega_z)$ is the vector of slow variables, θ and φ are fast variables, and the dot denotes a derivative with respect to time t.

The moment acting in the plane of the angle of nutation θ, is represented in the form

$$M_{xn} = M_{xn}^0 + \Delta M_{xn} \tag{1.2}$$

Here

$$M_{xn}^0 = -G\Delta z\sin\theta \tag{1.3}$$

$$\Delta M_{xn} = G\Delta y\cos\theta\cos\varphi + \Delta M_x\cos\varphi - \Delta M_y\sin\varphi + m^{\omega}\dot{\theta}$$

where M_{xn}^0 is the moment of the gravity force G corresponding to the unperturbed motion of the rigid body ($\Delta\mathbf{M} = 0$, $M_x^{\omega} = M_y^{\omega} = M_z^{\omega} = 0$, $\Delta y = 0$) and ΔM_{xn} is the perturbing moment.

We will consider the case, similar to the Lagrange case, and we will therefore assume that

$$M_{z1} \ll M_{xn}^0, \quad M_z \ll M_{xn}^0, \quad \Delta M_{xn} \ll M_{xn}^0$$

Then, after scaling of the perturbing functions in system (1.1) we obtain

$$\dot{x} = \varepsilon X(x,\varphi,\theta,\dot{\theta}), \quad \ddot{\theta} + F(x,\theta) = \varepsilon[f_1(x,\varphi,\theta) + f_2\dot{\theta}], \quad \dot{\varphi} = \Phi(x,\theta) \tag{1.4}$$

Here

$$\varepsilon X = (\varepsilon X_1, \varepsilon X_2), \quad \varepsilon X_1 = M_{z1}, \quad \varepsilon X_2 = M_z/I_z$$

$$F(x,\theta) = -M_{xn}^0 - F_1(x,\theta)F_2(x,\theta), \quad \Phi(x,\theta) = \omega_z - F_1(x,\theta)\cos\theta$$

$$\varepsilon f_1 = G\Delta y\cos\theta\cos\varphi + \Delta M_x\cos\varphi - \Delta M_y\sin\varphi, \quad \varepsilon f_2 = m^{\omega}$$

In system (1.4) the small parameter $\varepsilon > 0$ represents a small displacement of the centre of mass of the body with respect to the axis of dynamic symmetry (Δy), the values of the perturbing moments (ΔM_x, ΔM_y, ΔM_z) and of the dissipative moments (m^ω, m^{ω_z}).

2. THE EQUATIONS OF MOTION WITH RESPECT TO THE INTEGRAL MANIFOLD

For a system of the form (1.2) a procedure was developed in [7] for constructing the integral manifold, which enables one to investigate motions taking into account the lower-order resonance of the form $\Phi(x, \theta) \approx 0$ occurring in the system. To do this, system (1.4) is reduced to the singularly perturbed form

$$x' = \mu X(x, \varphi, \theta, \omega_\theta) \tag{2.1}$$

$$\mu\omega_\theta' = -F(x,\theta) + \mu^2[f_1(x,\varphi,\theta) + f_2\omega_\theta]$$

$$\mu\theta' = \omega_\theta, \quad \varphi' = \rho(x,\theta)$$

The prime denotes a derivative with respect to the slow time $\tau = \mu t$; $\mu = \sqrt{\varepsilon}$, $\omega_\theta = \dot{\theta}$, $\mu\rho(x, \theta) = \Phi(x, \theta)$.

When $\mu = 0$, we obtain the following degenerate system

$$x_0' = 0, \quad F(x_0, \theta_0) = 0, \quad \omega_{\theta_0} = 0, \quad \varphi_0' = \rho(x_0, \theta_0) \tag{2.2}$$

To separate the motions by the method of integral manifolds and to construct the corresponding asymptotic solutions, it is necessary to satisfy a number of conditions [6, 7], the most important of which are as follows:

1) the equation $F(x_0, \theta_0) = 0$ must have an isolated root $\theta_0 = \nu(x_0)$, where

$$\frac{\partial F}{\partial \theta}(x_0) > C_1 > 0 \tag{2.3}$$

where C_1 is a certain constant;

2) the matrix of the linear system, written with respect to the solutions of degenerate system (2.2), must have eigenvalues with negative real parts (see [7] for more detail).

The condition for the roots $\theta_0 = \nu(x_0)$ to exist in this problem is equivalent to the existence of stationary points in the Lagrange case and is certainly satisfied (see, for example, [12]).

Condition (2.3) for system (1.1) takes the form

$$\frac{\partial F}{\partial \theta} = -M_{xn}^{\theta}(\theta_0) - 3M_{xn}^{0}(\theta_0)\operatorname{ctg}\theta_0 + \frac{Q^2 + I_z^2\omega_z^2 - 2I_z\omega_z Q\cos\theta_0}{\sin^2\theta_0} > C_1 > 0 \quad \left(M_{xn}^{\theta} = \frac{\partial M_{xn}^{0}}{\partial \theta_0}\right)$$

and is always satisfied when $0 < \theta_0 \leqslant \pi/2$, which follows from the first expression of (1.3) ($\Delta z > 0$). When $\pi/2 < \theta_0 < \pi$ this condition is satisfied for fairly large values of $|Q|$ and $|\omega_z|$ and requires monitoring when carrying out analytic and numerical calculations.

The condition for the real parts of the corresponding matrix of the linear system to be negative, as was shown previously in [7], reduces to the existence of a dissipative moment $\varepsilon f_2 = m^\omega < -C_2 < 0$ ($C_2 = O(\varepsilon)$ is a certain positive constant), which acts in the plane of the angle of nutation. The value of this moment determines the velocity with which the trajectory of the system approaches to an integral manifold [6, 7]. It should be noted that, unlike the method of integral manifolds based on the results of the theory of singularly perturbed systems, developed by Tikhonov *et al.* [5], there is a modification of the method [6], in which a large value of the dissipative terms is not required (they can also be small of the order of ε).

The solution of degenerate system (2.2) does not give any interesting information on the motion with respect to the integral manifold since $x_0 = \text{const}$ and $\theta_0 = \text{const}$. To obtain interesting information we need to represent the solutions for the slow motion in the form of asymptotic series (for example, $\theta = \theta_0(\tau) + \mu\theta_1(\tau) + \mu^2 \ldots$), substitute them into system (2.1) and, using the standard procedure for constructing asymptotic solutions for singularly perturbed systems, determine the next terms of the series. By determining these solutions up to terms $O(\mu^2)$ (see [7] for more detail), we obtain for system (2.1)

$$\theta = \nu + \mu^2 f_1(x, \varphi, \nu)\left(\frac{\partial F}{\partial \nu}\right)^{-1}, \quad \omega_\theta = \mu^2 \frac{\partial \nu}{\partial x} X(x, \varphi, \nu) \tag{2.4}$$

where ν is the root of the equation $F(x, \nu) = 0$.

Relations (2.4), together with the differential equations for the variables x and φ, which remain at the same form as in system (2.1), describe the motion of the system with respect to the integral manifolds. For simplicity we have retained the same notation in series (2.4) as for the initial system (2.1).

3. REDUCTION OF THE SYSTEM OF EQUATIONS OF SLOW MOTION TO A STANDARD SYSTEM WITH A FAST PHASE

The motion of the system with respect to integral manifolds still remains fairly complex for analysis, since non-resonance sections of the motion (the phase φ rotates) and resonance sections exist, requiring different approaches in the investigation. Hence, we will reduce the slow-motion equations obtained to a system with a fast phase, bearing in mind the subsequent use of the well-developed averaging method to investigate it. We will first obtain the differential equations for the angle of nutation θ, by differentiating the first relation of (2.4) with respect to τ. We obtain

$$\theta' = \nu' + \mu^2 \frac{\partial f_1}{\partial \nu}\left(\frac{\partial F}{\partial \nu}\right)^{-1} \rho(x, \nu) + \mu^3 \ldots \tag{3.1}$$

where the derivative ν' is defined by the differential equation $F(x, \nu) = 0$ in the form

$$\nu' = -\mu\left(\frac{\partial F}{\partial \nu}\right)^{-1} \frac{\partial F}{\partial x} X(x, \varphi, \nu)$$

On the right-hand side of Eq. (3.1), we can put $\nu \approx \theta$ without violating the approximation employed.

The initial data for Eq. (3.1) with respect to the angle θ must belong to integral manifold (2.4). This will be satisfied for any value of θ if the variable Q is determined from the second equation of system (2.1), defined to terms of $O(\mu^2)$ rather than from the solution of the first equation of system (1.1) (here we have taken into account the fact that $\omega_\theta = O(\mu^2)$). Here

$$-F(x, \theta) + \mu^2 f_1(x, \varphi, \theta) = O(\mu^3)$$

Hence, taking into account the form of the functions F and f_1 and solving the quadratic equation for Q, we obtain

$$Q_{1,2} = \frac{I_z \omega_z}{2} \frac{1 + \cos^2\theta}{\cos\theta} \pm \left(\frac{I_z^2 \omega_z^2}{4} - M_{xn}^- \operatorname{ctg}\theta\right)^{1/2} \frac{\sin^2\theta}{\cos\theta} \tag{3.2}$$

where M_{xn}^- is the moment of M_{xn}, defined ignoring dissipation ($m^\omega = 0$). The plus sign in solution (3.2) corresponds to positive values of the angular velocity $\omega_z (\omega_z > 0)$, while the minus sign corresponds to negative values ($\omega_z < 0$).

Substituting expressions (3.2) into the condition $\varphi' = 0$, we obtain an expression for the resonance value of the angular velocity ω_z

$$\omega_z^r = \pm \omega / (1 - I_z)^{1/2}, \quad \omega = \left(-M_{xn}^- \operatorname{ctg}\theta\right)^{1/2} \tag{3.3}$$

By combining the differential equations for the variables θ (Eq. (3.1)) and the variables ω_z and φ (the second and fourth equations of (1.1)) and changing to the original time t, we obtain, in final form, a system with the fast phase φ, describing, up to terms $O(\mu^2) = O(\varepsilon)$, the motion with respect to the integral manifold

$$(\partial F / \partial \theta)\dot{\theta} = \left(3\omega_\psi \cos\theta - (1 + I_z)\omega_z\right) M_{yn} - M_z \omega_\psi \sin\theta$$

$$\dot{\omega}_z = M_z / I_z, \quad \dot{\varphi} = \omega_z - \omega_\psi \cos\theta \tag{3.4}$$

where

$$\partial F/\partial\theta = F^{(\theta)} = -M_{xn}^{\theta} + \omega_{\psi}^{2} + \left(I_z\omega_z - \omega_{\psi}\cos\theta\right)\left(I_z\omega_z - 2\omega_{\psi}\cos\theta\right)$$

$$\omega_{\psi} = \left(Q_{1,2} - I_z\omega_z\cos\theta\right)/\sin^2\theta,\quad M_{yn} = O(\varepsilon),\quad M_z = O(\varepsilon)$$

Representing system (3.4) in standard form, we obtain

$$\dot{y} = \varepsilon Y(y,\varphi),\quad \dot{\varphi} = \Delta_{\omega}(y) + \varepsilon\Omega(y,\varphi) \tag{3.5}$$

where

$$Y(y,\varphi) = (Y_1, Y_2)$$

$$Y_1 = -m^A\cos(\varphi + \varphi_A) + \omega_{1,2}\left(m^{\omega_z}\omega_z\,\mathrm{tg}\,\theta/F^{(\theta)} - m^{\omega}f_3\sin\theta\right)$$

$$Y_2 = \left(m^{\omega_z}\omega_z + \Delta M_z - G\Delta y\sin\theta\sin\varphi\right)/I_z$$

$$m^A = \left(\left(m_1^A\right)^2 + \left(m_2^A\right)^2\right)^{1/2}$$

$$\cos\varphi_A = -m_1^A/m^A,\quad \sin\varphi_A = m_2^A/m^A,\quad m_1^A = f_3\Delta M_y$$

$$m_2^A = f_3\left[\Delta M_x + G\Delta y\cos\theta - \left(G\Delta y\sin^2\theta\,\omega_{1,2}\right)/\cos\theta\right]$$

$$f_3 = \left[(1 + I_z)\omega_z - 3\omega_{1,2}\right]/F^{(\theta)},\quad \Delta_{\omega}(y) = \omega_z - \omega_{1,2}$$

$$\omega_{1,2} = \omega_{\psi}\cos\theta = I_z\omega_z/2 \pm \omega_a,\quad \omega_a = \left(I_z^2\omega_z^2/4 - M_{xn}^{-}\,\mathrm{ctg}\,\theta\right)^{1/2}$$

$$\Omega(y,\varphi) = \pm\Delta M_{xn}\,\mathrm{ctg}\,\theta/(2\omega_a)$$

Here $y = (\theta, \omega_z)$ is the new vector of the slow variables, $Y(y, \varphi)$ is a vector function, periodic in the phase with period 2π, and m^A and φ_A are certain generalized asymmetry parameters of the rigid body.

4. AVERAGED EQUATIONS FOR THE SLOW VARIABLES

To analyse the change in the slow variables during the non-resonance parts of the motion using the standard procedure [13], we will determine the averaged equations of the second approximation. The need to use the second approximation is due to the fact that the first approximation of the averaging method, as is well known, does not contain singularities related to the resonances which arise in the system. The determination of the second approximation enables us to obtain a more complete picture

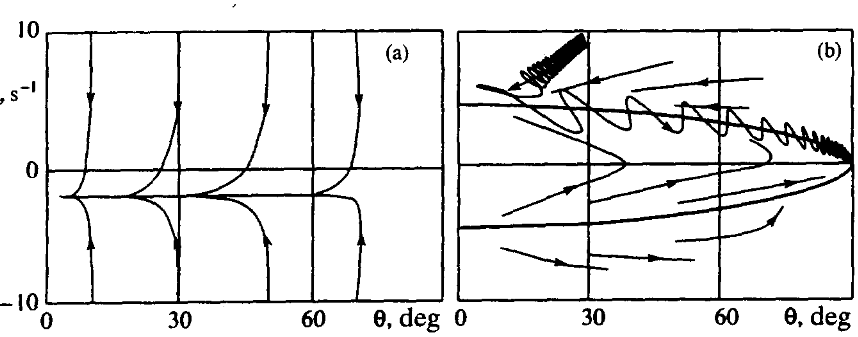

Fig. 1

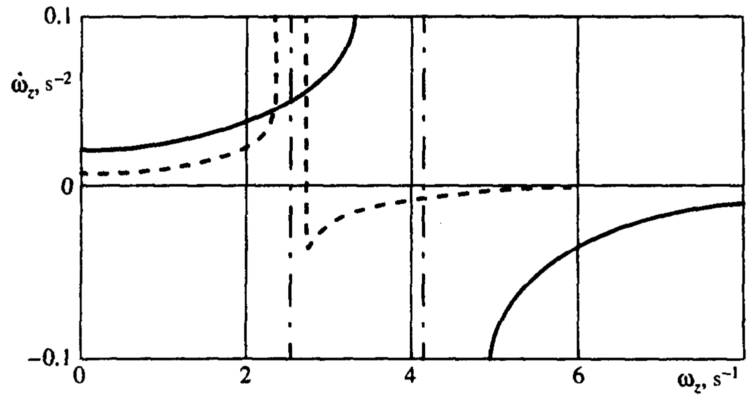

Fig. 2

of the motions in the system, in particular, to analyse resonance evolutionary effects. After using the averaging method for the slow variables of system (3.5) we obtain

$$\langle\dot{\theta}\rangle = \varepsilon A_1^{\theta} + \varepsilon^2 A_2^{\theta}, \quad \langle\dot{\omega}_z\rangle = \varepsilon A_1^{\omega} + \varepsilon^2 A_2^{\omega} \tag{4.1}$$

where

$$A_1^{\theta} = m^{\omega} f_3 \omega_{1,2} \sin\theta + m^{\omega_z}\omega_z\omega_{1,2}\,\mathrm{tg}\,\theta / F^{(\theta)}$$

$$A_2^{\theta} = \pm\frac{m^A}{4\omega_a\Delta_\omega\,\mathrm{tg}\,\theta}\left[(\Delta M_x + G\Delta y\cos\theta)\cos\varphi_A + \Delta M_y\sin\varphi_A\right] -$$

$$-\frac{\partial(m^A\cos\varphi_A)}{\partial\omega_z}\frac{G\Delta_y\sin\theta}{2I_z\Delta_\omega} + \left(1 \mp \frac{I_z^2\omega_z}{4\omega_a}\right)\frac{m^A}{2I_z\Delta_\omega^2}G\Delta y\sin\theta\cos\varphi_A -$$

$$-\frac{\partial\varphi_A}{\partial\theta}\frac{(m^A)^2}{2\Delta_\omega}, \quad A_1^{\omega} = \frac{m^{\omega_z}\omega_z + \Delta M_z}{I_z}$$

$$A_2^{\omega} = \frac{2m^A\omega_a\cos\varphi_A \mp \Delta M_y}{4I_z\omega_a\Delta_\omega}G\Delta y\cos\theta \mp \frac{m^A}{8I_z\omega_a\Delta_\omega^2}\Delta z\Delta y G^2\sin 2\theta\cos\varphi_A$$

In system (4.1) for the averaged variables, for simplicity we have retained the same notation as in system (3.4). A feature of the averaged equations (4.1) is the presence in the denominators of corresponding expressions for the frequency detuning Δ_ω, the value of which determines how close the trajectory is to the resonances considered. If the trajectory of the system approaches the resonance curve (3.3), the order of the terms on the right-hand sides of system (4.1) changes. In particular, if $\Delta_\omega = O(\varepsilon^{1/2})$ (the traditional quantity of the resonance zone [14]), the orders of the functions εA_1^{θ}, εA_1^{ω} and $\varepsilon^2 A_2^{\theta}$, $\varepsilon^2 A_2^{\omega}$ become the same, which follows from the expressions derived above. Consequently, the smaller the value of Δ_ω the greater the influence of the terms of the second approximation on the evolution of the variables θ and ω_z (the secondary resonance effect [8]).

System (4.1) is a second-order autonomous system. Possible motions in it can be conveniently analysed in the phase plane (ω_z, θ). The results of such an analysis for a statically stable Lagrange top ($\theta \in [0, \pi/2]$) are presented in Fig. 1 for the following initial data (we recall that G, I_z, m^{ω} and m^{ω_z} are dimensionless quantities normalized by the axial moment of inertia I)

$$G = 5\mathrm{m}^{-1}\mathrm{s}^{-2},\ I_z = 0.05,\ I = 40\ \mathrm{kgm}^2,\ \Delta z = 3\mathrm{m},\ m^{\omega} = -0.0075\ \mathrm{s}^{-1}$$

$$\Delta y = 0,\ m^{\omega_z} = -0.0063\ \mathrm{s}^{-1},\ \Delta M_z = 0.0125\ \mathrm{s}^{-2},\ \Delta M_y = 0\ \text{(Fig. 1a)}$$

$$\Delta y = -0.04\mathrm{m},\ m^{\omega_z} = 0,\ \Delta M_z = 0,\ \Delta M_y = -0.0063\ \mathrm{s}^{-2}\ \text{(Fig. 1b)}$$

Two basically different cases of perturbed motion of the body are illustrated: when the behaviour of the system in the phase plane $\omega_z = \omega_z(\theta)$ is determined solely by terms of the first approximation of the method of averaging ($A_1^\theta, A_1^\omega \neq 0$) and when $A_1^\omega = 0$ (when $m^{\omega_z} = 0$, $\Delta M_z = 0$), i.e. evolutionary effects with respect to the variable ω_z are determined by the second approximation A_2^ω.

In the first case, analysis of the first approximation (Fig. 1a) of the averaging method (taking into account the functions A_1^θ, A_1^ω only) in system (4.1) reduces to finding two singular points with coordinates $(-\Delta M_z/m^{\omega_z}, 0)$ and $(-\Delta M_z/m^{\omega_z}, \pi/2)$ in the (ω_z, θ) plane, where the first of these is stable and the second is unstable; in fact, the phase portrait (Fig. 1a) in this case corresponds to the motion of a symmetrical body. In the second case, the phase portrait of this system is considerably changed: the resonance curves $\omega_z^r(\theta)$ (3.3) (shown in Fig. 1b by the thick curve) divide the plane into three regions, and the evolution of the variables ω_z and θ is determined by the fact that the initial data $\omega_z(0)$ and $\theta(0)$ belong to the corresponding region and by the sign of the product $\Delta y \Delta M_y$, characterizing the asymmetry of the body. If $\Delta y \Delta M_y > 0$, the positive branch of the resonance curve $\omega_z^r(\theta) > 0$ is stable and the phase portraits converge to it. If $\Delta y \Delta M_y < 0$, the negative branch ($\omega_z^r(\theta) < 0$) is stable. When $\Delta y = \Delta M_y = 0$ the evolutionary motions of the system are determined by the first approximation of the averaging method (Fig. 1a), since $A_2^\theta, A_2^\omega = 0$.

For large angles of nutation ($\theta > 70°$) close to the resonance curve $\omega_z^r(\theta)$, it is possible for singular points to occur, the stability (or instability) of which is identical with the stability (or instability) of the corresponding branch (positive or negative) of the resonance curve. The occurrence of singular points close to the resonance values (the dash-dot curves) is illustrated in Fig. 2 for $\theta > 30°$ and $> 70°$ (the continuous and dashed curves $\dot\omega_z = \dot\omega_z(\omega_z)$ respectively) and is explained by the presence in the expression for the functions A_2^ω of denominators which depend on the squares of the resonance relations of the frequencies Δ_ω^2.

Hence, during the time when Δy, $\Delta M_y \neq 0$ the phase trajectory tends to one of the resonance curves and the further evolution of the system is determined by the behaviour of the system in the resonance region.

A detailed analysis of the motion of the system in the resonance region is outside the scope of this paper, since it requires another averaging procedure. However, we must note here that if the resonance is "internally" stable [8], the trajectory of the system, which remaining close to the resonance curve, is shifted along it in the direction of the singular point $\theta = \pi/2$, and tends to it asymptotically. A similar trajectory, shown in Fig. 1(b) by the thin curve, is obtained from system (3.5). Calculations were also carried out from the initial equations, which confirm the correctness of these results.

Hence, the approach described permits us to investigate fairly fully the pattern of resonance evolutionary effects along non-resonance parts of the motion of a rigid body in a case similar to the Lagrange case, which enables it to be used in a number of mechanics problems, for example, in the problem of the motion of an asymmetric body in a resisting medium.

This research was supported by the Russian Foundation for Basic Research (99-01-00477).

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Translated by R.C.G.